Exercise 57

Let $f(x) = \sqrt[3]{x}$.

(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.

(b) Show that $f'(0)$ does not exist.

(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 1.2.13.)

Solution

Use Equation 2.7.5 to find $f'(a)$, assuming $a \neq 0$.

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{(\sqrt[3]{x} - \sqrt[3]{a})\left(\sqrt[3]{x^2} + \sqrt[3]{a}\sqrt[3]{x} + \sqrt[3]{a^2}\right)} \\
&= \lim_{x \to a} \frac{1}{\left(\sqrt[3]{x^2} + \sqrt[3]{a}\sqrt[3]{x} + \sqrt[3]{a^2}\right)} \\
&= \frac{1}{\left(\sqrt[3]{a^2} + \sqrt[3]{a}\sqrt[3]{a} + \sqrt[3]{a^2}\right)} \\
&= \frac{1}{3\sqrt[3]{a^2}} \\
&= \frac{1}{3}a^{-2/3}
\end{aligned}$$

$f'(0)$ does not exist because

$$\begin{aligned}
f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\
&= \lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x} \\
&= \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} \\
&= \lim_{x \to 0} \frac{1}{x^{2/3}} \\
&= \infty
\end{aligned}$$

this limit does not exist. Note that $f'(0)$ represents the slope of $f(x)$ at $x = 0$. Since $f'(0) = \infty$, there is a vertical tangent at $x = 0$.

Below is a graph of $f(x) = \sqrt[3]{x}$ to illustrate the vertical tangent at $x = 0$.